

# Supersymmetric null-surfaces

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## Abstract

Single trace operators with the large R-charge in supersymmetric Yang-Mills theory correspond to the null-surfaces in  $AdS_5 \times S^5$ . We argue that the moduli space of the null-surfaces is the space of contours in the super-Grassmanian parametrizing the complex  $(2|2)$ -dimensional subspaces of the complex  $(4|4)$ -dimensional space. The odd coordinates on this super-Grassmanian correspond to the fermionic degrees of freedom of the superstring.

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# 1 Introduction.

In the AdS/CFT correspondence the single-trace operators of the large  $N$  supersymmetric Yang-Mills theory correspond to the single string states in the Type IIB theory on  $AdS_5 \times S^5$ . The single-trace operators have a form of the trace of the product of  $L$  elementary fields of the Yang-Mills theory. There is a certain subclass of these operators for which the corresponding string can be described semiclassically [1, 2, 3, 4]. For an operator to correspond to a classical dual string, one has to take  $L \rightarrow \infty$ . Also, in taking this limit, one has to arrange the elementary fields under the trace in such a way that the operator is “locally half-BPS” [5].

Let us explain what “locally half-BPS” means. The  $N = 4$  theory has six scalar fields  $\Phi_1 \dots, \Phi_6$ . Let us consider a complex combination  $Z = \Phi_5 + i\Phi_6$ . In the limit  $L \rightarrow \infty$  we have to require that each elementary field under the trace is of the form  $\phi_k = g_k \cdot Z$  where  $g_k$  is some element of the superconformal group  $SU(2, 2|4)$ . In other words, each elementary field is in the superconformal orbit of  $Z$ . Moreover we should have  $g_{k+1} = g_k + O(1/L)$ . Therefore instead of the discrete “chain” of the group elements  $g_1, \dots, g_L$  we have a continuous contour in the group manifold  $g(\sigma)$ , where  $\sigma = 2\pi k/L$ . In this “continuous limit” the anomalous dimension of the Yang-Mills operator becomes of the order  $\frac{\lambda}{L}$  plus the higher order corrections which are the series in  $\frac{\lambda}{L^2}$ . Moreover, the renormgroup flow defines a classical dynamical system on the space of contours  $g(\sigma)$  (see [6] and references therein). More precisely,  $g(\sigma)$  takes values not in the group manifold  $PSU(2, 2|4)$  itself but rather in the coset space which is  $PSU(2, 2|4)$  modulo the subgroup which acts on  $Z$  as a phase rotation. Therefore the renormgroup flow in the field theory defines a classical dynamical system on the space of loops  $g(\sigma)$  taking values in the supercoset  $Gr(2|2, 4|4) = U(2, 2|4)/(U(2|2) \times U(2|2))$ .

This supercoset has dimension  $(16|16)$ , sixteen even and sixteen odd coordinates. Therefore the “continuous” Yang-Mills operators are described by 16 even and 16 odd functions of one real variable. But 16 even and 16 odd functions also parametrize the phase space of the classical Type IIB superstring in  $AdS_5 \times S^5$ . Therefore, it is natural to conjecture that the classical dynamical system on the space of the locally half-BPS operators defined by the renormgroup flow is equivalent to the classical worldsheet dynamics of the Type IIB superstring.

Unfortunately we do not know any independent prescription which would tell us which string worldsheet corresponds to a given Yang-Mills operator. But the conjecture that the string sigma-model is equivalent to the classical renormgroup flow is nontrivial even without such a prescription. Indeed,

the equivalence of two dynamical systems is already a nontrivial statement. It was partially verified at the two loop level (in three different ways!) in [8, 9, 10].

Let us briefly review what happens at the one loop level, following [11, 12] (see also [13, 14] for a different approach). Consider the string worldsheet corresponding to a given Yang-Mills single trace operator composed of  $L$  elementary fields. The shape of the worldsheet depends on the coupling constant  $\lambda$ . When  $\frac{\lambda}{L^2} \rightarrow 0$  the worldsheet degenerates and becomes a null-surface. Moreover, this null-surface comes with a parametrization of the light rays. Therefore in the “continuous” limit the single trace operators correspond to the parametrized null surfaces. It turns out that the string worldsheet theory defines the structure of a Hamiltonian system on the space of parametrized null-surfaces. The definition of this Hamiltonian system goes as follows. Pick a parametrized null-surface. Consider a family of extremal surfaces depending on the parameter  $\epsilon$ , such that: 1) the limit when  $\epsilon \rightarrow 0$  is our null-surface and 2) the density of the conserved charges on the worldsheet in the limit  $\epsilon \rightarrow 0$  is proportional to  $\frac{1}{\epsilon} d\sigma$  where  $\sigma$  is the parametrizing function (see Section 3.3 of [12] for the precise formula). There are infinitely many such families, but for the purpose of our definition we can pick any one, satisfying these two properties. The deviation of this extremal surface from our null-surface is locally of the order  $\epsilon^2$ . But if we follow its evolution in the global AdS time, the deviation will accumulate. After the time interval of the order  $\Delta T \sim \frac{1}{\epsilon^2}$ , the deviation becomes of the order one, and our extremal surface will locally approximate (with the accuracy  $\sim \epsilon^2$ ) another null-surface. This determines the evolution on the space of null-surfaces.

An important point is that this definition does not depend on the choice of the family converging to the null-surface. If we pick two different world-sheets approximating the same parametrized null-surface, they will “oscillate” around each other, but the deviation between them will not accumulate in time. Therefore, different approximating surfaces determine the same “slow evolution” on the space of null-surfaces.

The space of parametrized null-surfaces is identified with the space of pairs of functions  $Y : S^1 \rightarrow \mathbf{C}^{2+4}$ ,  $Z : S^1 \rightarrow \mathbf{C}^6$  such that  $|Y(\sigma)|^2 = |Z(\sigma)|^2 = 2$  and  $(Y(\sigma), Y(\sigma)) = (Z(\sigma), Z(\sigma)) = 0$  and  $(\bar{Y}, \partial_\sigma Y) = (\bar{Z}, \partial_\sigma Z)$ . The one-loop anomalous dimension corresponds to the Hamiltonian of the slow evolution:

$$\Delta = \frac{1}{16\pi^2} \frac{\lambda}{(L/2\pi)} \int_0^{2\pi} d\sigma ((\partial_\sigma \bar{Z}, \partial_\sigma Z) - (\partial_\sigma \bar{Y}, \partial_\sigma Y)) \quad (1)$$

This space is a  $U(1)$ -bundle over a submanifold of the loop space of the product of two Grassmanians:

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \quad (2)$$

It consists of the loops satisfying certain integrality condition which corresponds to the cyclic invariance of the trace.

In this paper we will argue that the fermionic degrees of freedom on the worldsheet parametrize in the ultrarelativistic limit the odd directions of the supercoset space. Just as the fast-moving bosonic string corresponds to a contour in the product of two Grassmanians (2) the superstring defines a contour in

$$\frac{U(2,2|4)}{U(2|2) \times U(2|2)}$$

Turning on the fermionic degrees of freedom of the null-surface corresponds on the field theory side to considering operators with insertions of the fermions and the field strength.

Dynamical systems on supersymmetric coset spaces were studied in the recent papers [16].

The null-surface perturbation theory was previously studied in a closely related context in [15].

## 2 Single trace operators with large spin.

The single trace operators are the operators of the form  $\text{tr } \phi_1 \phi_2 \cdots \phi_n$  where  $\phi_1, \dots, \phi_n$  are the fundamental fields. The one loop anomalous dimension for all the single trace operators in the  $\mathcal{N} = 4$  Yang-Mills theory was computed in [17, 18]. For the one loop computation, each fundamental field under the trace can be considered a free field, and therefore it transforms in the singleton representation of the superconformal group. The one loop anomalous dimension corresponds to the hermitean operator (interaction Hamiltonian) acting on the trace of the product of the free fields. It was shown in [18] that this operator commutes with the generators of the superconformal group. In the planar limit the interaction is a sum of the pairwise interactions of all the fundamental fields under the trace which are next to each other in the product. In other words, the one loop anomalous dimension of the operator  $\text{tr } \phi_1 \phi_2 \cdots \phi_n$  is given by the sum of the diagrams involving  $\phi_1$  and  $\phi_2$ , diagrams involving  $\phi_2$  and  $\phi_3$  and so on. The one loop interaction preserves the number of the fundamental fields under the trace.

## 2.1 Coherent states.

The “continuous” approach to the computation of the anomalous dimension of the single trace operator was proposed in [6]. This approximation is useful when the number of the fields under the trace is very large. This approach (as we understand it) relies on the existence of the special set of vectors in the singleton representation. This set of vectors is obtained in the following way. Consider the free  $\mathcal{N} = 4$  theory on  $\mathbf{R} \times S^3$ . The vacuum is conformally invariant. Let us act on the vacuum by the creation operator of the free boson  $Z(x) = \phi_5(x) + i\phi_6(x)$  integrated over  $S^3$ . We will get a state which we will call  $\psi_1$ . This state is not invariant under  $PSU(2, 2|4)$ . For any group element  $g \in PSU(2, 2|4)$  we will denote  $\psi_g = g.\psi_1$ . We will call the states of the form  $g.\psi_1$  *coherent*.

It is important to identify the subgroup  $H \subset PSU(2, 2|4)$  which acts on  $\psi_1$  by rotating its phase. Let us first discuss the bosonic part of  $H$ . The bosonic part of  $PSU(2, 2|4)$  is<sup>2</sup>  $SU(2, 2) \times SU(4)$ . And the bosonic part of  $H$  consists of the shift of time and the isometries of  $S^3$ , which together form  $S(U(2) \times U(2)) = (SU(2) \times SU(2) \times U(1))/\mathbf{Z}_2$ , plus the subgroup  $S(U(2) \times U(2))$  of the R-symmetry  $SU(4)$  which preserves the direction of  $Z(x)$  in the isotopic space:

$$H_{rd} = S(U(2) \times U(2)) \times S(U(2) \times U(2)) \quad (3)$$

This is (modulo the global structure) the even part of the supergroup:

$$H = PS(U(2|2) \times U(2|2)) \quad (4)$$

This supergroup can be understood as the central extension of  $PSU(2|2) \times PSU(2|2)$  by the  $U(1)$  which we call  $U(1)_c$ , and another  $U(1)$  generated by the “grading element  $d$ ” entering as a semidirect product:

$$PS(U(2|2) \times U(2|2)) = U(1)_d \ltimes (PSU(2|2) \times PSU(2|2)) \ltimes U(1)_c \quad (5)$$

Here  $\ltimes$  denotes the semidirect product<sup>3</sup>; if  $M$  and  $N$  are two groups then  $G = M \ltimes N$  means that  $N$  is a normal subgroup in  $G$  and  $G/N = M$ . The

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<sup>2</sup>The global structure of the superconformal group is not very important here because the center of the superconformal group is a subgroup of  $H$ . We will consider the global issues in Section 3.

<sup>3</sup>In the original version of this paper we wrote incorrectly that the two factors  $U(1)$  enter through the direct product. In fact this is a semi-direct product. The most economical notation for  $H$  is  $PS(U(2|2) \times U(2|2))$ . We realized the mistake studying the recent works [19, 20, 21]. If there were two free factors  $U(1)$  then the classical superstring in  $AdS_5 \times S^5$  would have two series of local conserved charges; but in fact there is only one from  $U(1)_d$ .

purely bosonic part of (5) contains both  $U(1)_d$  and  $U(1)_c$  as free factors. But once we turn on the odd coordinates of the supergroup,  $U(1)_d$  does not commute with the fermionic generators of  $(PSU(2|2) \times PSU(2|2)) \ltimes U(1)_c$  and also  $PSU(2|2) \times PSU(2|2) \subset U(1)_d \ltimes (PSU(2|2) \times PSU(2|2)) \ltimes U(1)_c$  is not a subgroup because the anticommutator of two odd elements in  $PSU(2|2)$  generally speaking has a component in  $U(1)_c$ . In fact  $H$  can be defined as the centralizer of  $U(1)_c$  in  $PSU(2, 2|4)$ , which makes the contact with [19, 20, 21]. In the plane wave language  $U(1)_c$  corresponds to the light-like Killing vector field  $\frac{\partial}{\partial x^-}$ , and  $U(1)_d$  corresponds to the other Killing vector field  $\frac{\partial}{\partial x^+}$  which is not lightlike. The coordinate  $x^+$  is considered a time coordinate in the plane wave language. The subgroup  $U(1)_d$  does not commute with the fermionic part of  $(PSU(2|2) \times PSU(2|2)) \ltimes U(1)_c$  because the generators of the supersymmetry in the plane wave background are time-dependent [22, 23, 24].

Therefore the coherent states are parametrized by the points of the coset space

$$Gr(2|2, 4|4) = \frac{PSU(2, 2|4)}{PS(U(2|2) \times U(2|2))} = \frac{U(2, 2|4)}{U(2|2) \times U(2|2)} \quad (6)$$

These states generate the singleton representation. It is then conjectured that in the limit of the large number of fields the “semiclassical” states are the decomposable tensors of the form

$$\psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_n} \quad (7)$$

where the difference between  $g_k$  and  $g_{k+1}$  is of the order  $\frac{1}{n}$ . In the continuum limit  $n \rightarrow \infty$ , the number of the site  $k$  becomes a continuous parameter  $\sigma = k/n$ , and the evolution of the state is approximated by the classical evolution of the contour  $g(\sigma)$ . The classical Hamiltonian is the matrix element

$$H_{cl}[g(\sigma)] = (\bar{\psi}_{g_1} \otimes \bar{\psi}_{g_2} \otimes \cdots \otimes \bar{\psi}_{g_n}, H_{int} \psi_{g_1} \otimes \psi_{g_2} \otimes \cdots \otimes \psi_{g_n}) \quad (8)$$

The symplectic structure is  $\Omega = \int d\sigma \Omega(\sigma)$  where  $\Omega(\sigma)$  is the differential of the one-form

$$\alpha(\sigma) = (\bar{\psi}_{g(\sigma)}, d\psi_{g(\sigma)}) \quad (9)$$

This data defines a dynamical system on the space of contours in the super-Grassmanian (6). The interaction Hamiltonian of [18] involves only the pairs of neighbors in the product. Therefore the continuous Hamiltonian should be a local functional of the contour. Moreover, one can see that it contains not more than two derivatives  $\partial_\sigma$ . Therefore the value of the Hamiltonian

on the contour should be given by the value of the classical action of the free particle on the super-Grassmanian which has this contour as a trajectory<sup>4</sup>.

It is useful to write down these coherent states more explicitly. The space of one-particle states of the free scalar field theory can be identified with the space of the positive-frequency solutions of the free field equations. We will describe a family of positive-frequency solutions parametrized by the points of the coset  $\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)}$ . This coset is parametrized by two complex lightlike vectors  $Y$  and  $Z$ ,  $(\bar{Y}, Y) = (\bar{Z}, Z) = 2$ ,  $(Y, Y) = (Z, Z) = 0$ , modulo independent phase rotations of  $Y$  and  $Z$ . The manifold of the complex lightlike vectors  $Y$  in  $\mathbf{C}^{2+4}$  consists of two connected components. Those  $Y$  which can be rotated by  $SO(2, 4)$  to  $(1, -i, 0, 0, 0, 0)$  belong to the first component, and those which can be rotated to  $(1, i, 0, 0, 0, 0)$  belong to the second component. We need only the first component. Given  $Y$  and  $Z$ , let us consider the positive frequency wave of the field  $Z_1\Phi_1 + \dots + Z_6\Phi_6$  parametrized by  $Y$ :

$$f_{[Y]}(\tau, \mathbf{n}) = \frac{1}{Y_{-1} \cos \tau + Y_0 \sin \tau - (\mathbf{Y}, \mathbf{n})} \quad (10)$$

$$\left( \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \mathbf{n}^2} \right) f_{[Y]} = -f_{[Y]}$$

Here  $(\tau, \mathbf{n})$  are the coordinates on  $\mathbf{R} \times S^3$ ,  $\tau$  is parametrizing  $\mathbf{R}$  and  $\mathbf{n} = (n_1, n_2, n_3, n_4)$  is the unit vector parametrizing  $S^3$ . This defines a state  $\psi_{[Y],[Z]}$ .

The super-Grassmanian (6) is the (4,2,2) analytic superspace of [25, 26, 27]. The bosonic coset space  $\frac{SO(2,4)}{SO(2) \times SO(4)} \simeq \frac{SU(2,2)}{S(U(2) \times U(2))}$  was discussed in the context of the AdS/CFT correspondence in [28], where it was identified as the moduli space of timelike geodesics in  $AdS_5$ . This coset space is the future tube of the Minkowski spacetime (see [28] and references therein, and [29, 30] for the mathematical background). The motion of a particle in AdS space was also studied in [31]. An interesting technique for obtaining the singleton representation from dynamics in higher dimensional spaces was developed in [32]; in the next section we will use an approach related to the ideas of [32] to study spinors in AdS space. The general theory of the coherent states of the type discussed in this section was developed in [29].

The singleton representation of  $SO(2, 4)$  is not square integrable (see [33] for a recent discussion of this fact, and references therein). Therefore our system of coherent states does not resolve the identity.

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<sup>4</sup>The contour can be an arbitrary trajectory, not necessarily satisfying the equations of motion.

## 2.2 Anomalous dimension.

The coherent states of the spin chain generally speaking do not have a definite energy in the  $\lambda = 0$  theory. In fact the corresponding operators are superpositions of operators with different engineering dimensions. In this situation we can define the one-loop anomalous dimension in the following way. The superconformal group is actually  $\widehat{PSU}(2, 2|4)$  — the universal covering of  $PSU(2, 2|4)$ . The universal covering has a center  $C = \mathbf{Z}$ . Let  $c$  denotes the generator of the center. In the free field theory  $c = 1$ , but it acts nontrivially in the interacting theory:

$$c = e^{i\Delta} \quad (11)$$

In perturbation theory  $\Delta$  is expanded in powers of  $\lambda$ . It starts with the term linear in  $\lambda$ , which is the one-loop anomalous dimension. It is obvious from this definition that the one-loop anomalous dimension commutes with the superconformal group.

At the one-loop level the action of the center is the sum of the contributions of the pairwise interactions of the nearest neighbors of the parton chain. Each pairwise interaction separately commutes with the superconformal group. Therefore it is enough to consider the case when the pair of nearest neighbors is:

$$\dots \otimes Z(0) \otimes (Z(0) + a\mathcal{O}_1 + a^2\mathcal{O}_2 + \dots) \otimes \dots \quad (12)$$

Here  $a$  is the lattice spacing and  $\mathcal{O}_n$  are some combinations of elementary fields. For the continuous operators

$$\mathcal{O}_1 = \alpha_I \Phi^I(0) + \beta^\mu \partial_\mu Z(0) \quad (13)$$

where  $\alpha_I$  ( $I = 1, 2, 3, 4$ ) and  $\beta^\mu$  are some complex coefficients. We need to compute the expectation value of the interaction Hamiltonian in the coherent state:

$$\langle H \rangle = (\overline{Z}(0) \otimes (\overline{Z}(0) + a\overline{\mathcal{O}}_1 + \dots), H Z(0) \otimes (Z(0) + a\mathcal{O}_1 + \dots))$$

Notice that  $H.Z(0) \otimes Z(0) = 0$  because  $Z(0) \otimes Z(0)$  is the vacuum (“the BMN vacuum”). Therefore

$$\langle H \rangle = a^2 (\overline{Z}(0) \otimes \overline{\mathcal{O}}_1, H Z(0) \otimes \mathcal{O}_1) + o(a^2) \quad (14)$$

All that we need in the continuum limit is the leading term. The action of  $H$  was computed in [18]. We will now briefly review some results of



[18]. The tensor product of two supersingletons  $V_F$  of  $PSU(2,2|4)$  is a reducible representation. It decomposes into the direct sum of irreducible representations of  $PSU(2,2|4)$ :

$$V_F \otimes V_F = \bigoplus_{j=0}^{\infty} V_j \quad (15)$$

The action of  $H$  is the sum of the projectors  $P_j$  on  $V_j$ , with the coefficients depending on  $j$ . The coefficient of  $P_0$  is zero, and the coefficient of  $P_1$  is  $\frac{\lambda}{4\pi^2}$ . The BMN vacuum  $Z(0) \otimes Z(0)$  belongs to  $V_0$ . What can we say about  $Z(0) \otimes \mathcal{O}_1$  where  $\mathcal{O}_1$  is given by (13)? One can see that the symmetric part  $Z(0) \otimes \mathcal{O}_1 + \mathcal{O}_1 \otimes Z(0)$  belongs to  $V_0$  and the antisymmetric part  $Z(0) \otimes \mathcal{O}_1 - \mathcal{O}_1 \otimes Z(0)$  belongs to  $V_1$ . This means that:

$$(\bar{\Psi}, H.\Psi) = (\bar{\Psi}, \frac{\lambda}{8\pi^2} \sum_{l=1}^L (1 - P_{l,l+1})\Psi) + \dots \quad (16)$$

where dots denote terms subleading in the continuous limit. To compute the right hand side, we need the scalar product  $(\bar{\psi}_{[Y],[Z]}, \psi_{[Y'],[Z']}) = \frac{(\bar{Z}, Z')}{(Y, Y')}$ . Therefore the one-form (9) is  $(\bar{Y}, dY) - (\bar{Z}, dZ)$ , and the classical Hamiltonian  $(\bar{\Psi}, H.\Psi)$  is given by (1).

### 3 Fast moving superstrings.

#### 3.1 Anomalous dimension as a deck transformation.

$\widetilde{AdS}$  space is the universal covering space of the hyperboloid. The center of  $\widetilde{PSU}(2,2|4)$  acts as a deck transformation exchanging the sheets. We can visualize the action of this deck transformation on the string phase space in the following way. Let us replace  $AdS_5$ , which is the covering space of the hyperboloid, by the hyperboloid itself  $AdS_5/\mathbf{Z}$ . Let us formally consider the string as living on  $(AdS_5/\mathbf{Z}) \times S^5$ . Let us pick a point  $x$  on the string worldsheet  $\Sigma$ . Consider a neighborhood of  $x$  in  $(AdS_5/\mathbf{Z}) \times S^5$  which is simply connected. For example, we can pick as such a neighborhood a set of points which are within the distance  $R/2$  from  $x$ , where  $R$  is the radius of  $AdS_5$ . Let  $B$  denote such a neighborhood. Consider the part of the string worldsheet which is inside  $B$  (that is,  $B \cap \Sigma$ ). One can see that  $B \cap \Sigma$  consists of several sheets, which can be enumerated. These sheets are two-dimensional, so we can think of them as cards;  $B \cap \Sigma$  is then a deck of cards. Let  $x$  belong to the sheet number  $n$ , then we can draw a path on

$\Sigma$  starting at  $x$ , winding once on the noncontractible cycle in  $AdS_5/\mathbf{Z}$  and then ending on the sheet number  $n + 1$ . Let  $\Sigma_n$  denote the sheet number  $n$ . The coordinate distance between  $\Sigma_n$  and  $\Sigma_{n+1}$  is of the order  $\epsilon^2$ . The deck transformation maps  $\Sigma_n$  to  $\Sigma_{n+1}$ ,  $\Sigma_{n+1}$  to  $\Sigma_{n+2}$  and so on. This determines the action of the discrete group  $\mathbf{Z}$  on the string phase space, which should be identified with the action of the center of the superconformal group on the field theory side.

In this paper we consider only the one-loop approximation. Therefore we need the action of  $\mathbf{Z}$  to the order  $\epsilon^2$ . At this order the deck transformation can be interpreted as the slow evolution. Indeed, let us replace  $n$  with the continuous parameter  $t = n\epsilon^2$ . For every  $n$  the corresponding sheet  $\Sigma_n$  is close to some null-surface which we denote  $\Sigma(0)^{(t)}$ ; this null-surface is defined for each  $t$ ; there is an ambiguity in the definition of  $\Sigma(0)^{(t)}$  but it is of the order  $\epsilon^2$ . Therefore, in the limit  $\epsilon^2 \rightarrow 0$  the deck transformations define a one-parameter family of transformations of the null-surfaces. This slow evolution of the null surfaces was studied in [12] but only in the bosonic sector. In this section we will turn on the fermionic degrees of freedom.

### 3.2 Supersymmetric null surfaces.

The worldsheet theory for the superstring in  $AdS_5 \times S^5$  can be formulated as a sigma-model with the target space the supercoset

$$M = \frac{PSU(2,2|4)}{SO(1,4) \times SO(5)} \quad (17)$$

In the ultrarelativistic limit the string worldsheet becomes a parametrized null-surface. The parametrized null-surfaces correspond to the loops in the coset space

$$\frac{SO(2,4)}{SO(2) \times SO(4)} \times \frac{SO(6)}{SO(2) \times SO(4)} \simeq \frac{SU(2,2)}{S(U(2) \times U(2))} \times \frac{SU(4)}{S(U(2) \times U(2))}$$

This is the bosonic part of the story. Let us study the effect of the fermionic degrees of freedom.

We will apply the Green-Schwarz approach for the string in  $AdS_5 \times S^5$  developed in [34], but using a different representation of the gamma-matrices. Following [35] we will represent the spinors in the space  $AdS_5 \times S^5$  as restrictions of spinors in the flat space  $\mathbf{R}^{2+10}$ . We prefer this representation, because the covariantly constant spinors on  $AdS_5 \times S^5$  correspond in this picture to the constant spinors in the flat space. This also agrees with the philosophy of [32]. Consider the embeddings  $AdS_5 \subset \mathbf{R}^{2+4}$  with coordinates

$X_{-1}, X_0, \dots, X_4$  and  $S^5 \subset \mathbf{R}^6$  with coordinates  $X_5, \dots, X_{10}$ . Consider the twelve-dimensional chiral spinors  $\Psi$  of  $SO(2, 10)$ . These twelve-dimensional spinors are sections of the spinor bundle over  $\mathbf{R}^{2+10}$ , which is a trivial bundle (the product  $\mathbf{R}^{2+10} \times \mathbf{C}^{32}$ ). Let us restrict this bundle to  $AdS_5 \times S^5 \subset \mathbf{R}^{2+10}$ . This restriction would be a trivial bundle  $AdS_5 \times S^5 \times \mathbf{C}^{32}$ . It turns out that the spinor bundle on  $AdS_5 \times S^5$  can be realized as a subbundle of this bundle. This subbundle is the image of the projector  $\frac{1}{2}(1 + \Gamma^A \Gamma^S)$  where  $\Gamma^A$  is the  $\Gamma$ -matrix corresponding to the vector of the length square  $-1$  orthogonal to  $AdS_5$  in  $\mathbf{R}^{2+4}$  and  $\Gamma^S$  corresponds to the unit vector orthogonal to  $S^5$  in  $\mathbf{R}^6$ . Therefore a section of the spinor bundle on  $AdS_5 \times S^5$  can always be represented in the form:

$$\psi = \frac{1}{2} (1 + \Gamma^A \Gamma^S) \Psi_{++} \quad (18)$$

where  $\Psi_{++}$  satisfies:

$$\begin{aligned} \Gamma_{-1} \Gamma_0 \cdots \Gamma_4 \Psi_{++} &= i \Psi_{++} \\ \Gamma_5 \cdots \Gamma_{10} \Psi_{++} &= i \Psi_{++} \end{aligned} \quad (19)$$

This condition means that  $\Psi_{++}$  is the product of the positive chirality spinor of  $SO(2, 4)$  and the positive chirality spinor of  $SO(6)$ . We will choose the  $\Gamma$ -matrices to be real. We will denote  $\rho_{SU(2,2)}$  the space of the positive chirality spinor representation of  $SO(2, 4)$  (the fundamental of  $SU(2, 2)$ ) and  $\rho_{SU(4)}$  the space of the positive chirality spinor representation of  $SO(6)$  (the fundamental of  $SU(4)$ ). Therefore  $\Psi_{++} \in \rho_{SU(2,2)} \otimes \rho_{SU(4)}$ .

The worldsheet degrees of freedom are two Majorana-Weyl spinors  $\theta^1(\tau, \sigma)$ ,  $\theta^2(\tau, \sigma)$ . We will parametrize them by a single complex  $\Psi_{++}$ :

$$\begin{aligned} \theta^1 &= (1 + \Gamma^A \Gamma^S) \text{Re}(\Psi_{++}) \\ \theta^2 &= (1 + \Gamma^A \Gamma^S) \text{Im}(\Psi_{++}) \end{aligned} \quad (20)$$

The  $\gamma$ -matrices of [34] are related to the gamma-matrices in the tangent space to  $AdS_5 \times S^5$ :

$$i\gamma^a = \Gamma^a \hat{F}_A, \quad \gamma^{a'} = \Gamma^{a'} \hat{F}_S, \quad \hat{F}_A \theta^I = \hat{F}_S \theta^I \quad (21)$$

Here  $\hat{F}_A$  is the product of the five gamma-matrices  $\Gamma_a$  tangent to  $AdS_5$  and  $\hat{F}_S$  is the product of gamma-matrices tangent to  $S^5$ . Also, for any vector  $v$  in the tangent space to  $AdS_5 \times S^5$  we will denote:

$$\hat{v} = \Gamma_\mu v^\mu$$

Let us introduce a notation:

$$\bar{\theta} = \theta^T \Gamma_{-1} \Gamma_0 \quad (22)$$

where the superindex  $T$  denotes the transposition. In this definition we assume that we have chosen the gamma-matrices so that  $\Gamma_{-1}$  and  $\Gamma_0$  are antisymmetric, and  $\Gamma_1, \dots, \Gamma_{10}$  are symmetric. The definition (22) is for a real spinor  $\theta$ . For a complex combination  $\theta_1 + i\theta_2$  we define

$$\overline{\theta_1 + i\theta_2} = \bar{\theta}_1 - i\bar{\theta}_2 \quad (23)$$

This notation allows us to write an explicit formula for the linear map from the tensor product of two chiral spinor bundles to the vector bundle:

$$\theta_1 \otimes \theta_2 \mapsto j, \quad j^\mu = \bar{\theta}_1 \Gamma^A \Gamma^\mu \theta_2 \quad (24)$$

We will use the same notation for the conjugate of  $\Psi_{++}$ :

$$\overline{\Psi_{++}} = \Psi_{++}^{*T} \Gamma_{-1} \Gamma_0 \quad (25)$$

where  $\Psi_{++}^*$  means the complex conjugate of  $\Psi_{++}$  (notice that  $\Psi_{++}$  has to be a complex spinor).

The covariant derivative modified by the Ramond-Ramond field strength is:

$$\mathcal{D}_i(\theta^1 + i\theta^2) = \left[ D_i + \frac{1}{4} i(\hat{F}_A - \hat{F}_S) \Gamma_i \right] (\theta^1 + i\theta^2) \quad (26)$$

The main advantage of considering ten-dimensional spinors as restrictions of twelve-dimensional spinors is a simple form of the covariant derivative:

$$\mathcal{D}_i [(1 + \Gamma^A \Gamma^S) \Psi_{++}] = (1 + \Gamma^A \Gamma^S) \partial_i \Psi_{++} \quad (27)$$

This means that covariantly constant spinors correspond to constant  $\Psi_{++}$ .

In this paper we will restrict ourselves to the study of the configurations near  $\theta^I = 0$ ; we will only keep the terms of the lowest order in  $\theta^I$  (terms quadratic in  $\theta^I$  in the action). With this restriction, the kappa-transformations are:

$$\delta_k \theta^1 = \widehat{\partial_+ x} k^1 \quad (28)$$

$$\delta_k \theta^2 = \widehat{\partial_- x} k^2 \quad (29)$$

and the equations of motion for fermions are:

$$\widehat{\partial_+ x} \mathcal{D}_- \theta^1 = 0 \quad (30)$$

$$\widehat{\partial_- x} \mathcal{D}_+ \theta^2 = 0 \quad (31)$$

They imply that there are spinors  $\eta^1, \eta^2$  such that

$$\mathcal{D}_- \theta^1 = \widehat{\partial_+ x} \eta^1 \quad (32)$$

$$\mathcal{D}_+ \theta^2 = \widehat{\partial_- x} \eta^2 \quad (33)$$

Doing the kappa-transformation with the parameters  $k^1, k^2$  such that  $D_- k^1 = -\eta^1$  and  $D_+ k^2 = -\eta^2$  we are left with

$$\mathcal{D}_- \theta^1 = \mathcal{D}_+ \theta^2 = 0 \quad (34)$$

There are some “residual” kappa-transformations which preserve this condition.

For the fast moving string, we choose the coordinates  $\tau, \sigma$  so that  $g_{\tau\tau} = \epsilon^2$ ,  $g_{\sigma\sigma} = -1$ ,  $g_{\tau\sigma} = 0$ . The equations of motion in the “complex” notations is:

$$\mathcal{D}_\tau(\theta^1 + i\theta^2) - \epsilon (\mathcal{D}_\sigma(\theta^1 + i\theta^2))^* = 0 \quad (35)$$

In terms of  $\Psi_{++}$ :

$$\partial_\tau \Psi_{++} - \epsilon \Gamma^A \Gamma^S (\partial_\sigma \Psi_{++})^* = 0 \quad (36)$$

What happens when  $\epsilon \rightarrow 0$ ? We have  $\partial_\tau \Psi_{++} = 0$ , thus  $\Psi_{++}$  is constant on the light rays forming the null surface. Let us study the residual kappa-transformations. Let  $\partial_\tau x_A$  be the AdS-component of the tangent vector to the null-geodesic, and  $\partial_\tau x_S$  be the component in the tangent space to the sphere. The following kappa-transformation with the constant parameter  $K_{++}$  leaves  $\Psi_{++}$  constant along the light rays:

$$\begin{aligned} \delta_K [(1 + \Gamma^A \Gamma^S) \Psi_{++}] &= (\widehat{\partial_\tau x_A} + \widehat{\partial_\tau x_S})(1 + \Gamma^A \Gamma^S) \Gamma^A K_{++} = \\ &= (1 + \Gamma^A \Gamma^S) (\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S) K_{++} \end{aligned}$$

In other words,

$$\delta_K \Psi_{++} = (\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S) K_{++} \quad (37)$$

The right hand side is constant on the light ray, because  $\widehat{\partial_\tau x_A} \Gamma^A$  is the rotation in the equatorial plane of  $AdS_5$  and  $\widehat{\partial_\tau x_S} \Gamma^S$  is the rotation in the equatorial plane of  $S^5$ . These generators of rotations are very useful. In the spinor language the equator of  $AdS_5$  corresponds to the 2-plane  $L_A \subset \rho_{SU(2,2)}$  which is defined as the subspace on which the rotation in the equatorial plane acts with the eigenvalue  $+i$ . And the equatorial plane of  $S^5$  corresponds to the 2-plane  $L_S \subset \rho_{SU(4)}$ . We have a decomposition:

$$\rho_{SU(2,2)} \otimes \rho_{SU(4)} = (L_A \otimes L_S) \oplus (L_A^\perp \otimes S_S^\perp) \oplus (L_A \otimes S_S^\perp) \oplus (L_A^\perp \otimes L_S) \quad (38)$$

The kappa-transformation (37) then implies that the component of  $\Psi_{++}$  which belongs to  $(L_A \otimes L_S) \oplus (L_A^\perp \otimes L_S^\perp)$  is a pure gauge; it can be gauged away. This means that we can choose  $\Psi_{++}$  to satisfy

$$(\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S) \Psi_{++} = 0 \quad (39)$$

or equivalently  $(\widehat{\partial_\tau x_A} - \widehat{\partial_\tau x_S}) \theta^I = 0$ . In other words  $\Psi_{++}$  can be brought into the form

$$\Psi_{++} = \tilde{\eta} \otimes \phi + \eta^* \otimes \tilde{\phi}^* \quad (40)$$

where  $\phi, \tilde{\phi}^*$  are in the fundamental representation of  $SU(4)$  and  $\eta^*, \tilde{\eta}$  are in the fundamental representation of  $SU(2, 2)$  and  $\phi \in L_S$ ,  $\tilde{\phi}^* \in L_S^\perp$ ,  $\eta^* \in L_A$ ,  $\tilde{\eta} \in L_A^\perp$ . Let us introduce an orthonormal basis in  $\rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$ , according to the decomposition  $L_A \oplus L_A^\perp \oplus L_S^* \oplus L_S^{\perp*}$ . In this basis, the antihermitean matrix

$$\begin{bmatrix} 0 & 0 & 0 & \eta^* \otimes \tilde{\phi}^* \\ 0 & 0 & \tilde{\eta} \otimes \phi & 0 \\ 0 & -\phi^* \otimes \tilde{\eta}^* & 0 & 0 \\ -\tilde{\phi} \otimes \eta & 0 & 0 & 0 \end{bmatrix} \quad (41)$$

defines an infinitesimal variation of the 4-dimensional plane  $L_A \oplus L_S^*$  in the 8-dimensional space  $\rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$ , such that the variation of  $L_A$  goes outside  $\rho_{SU(2,2)}$  into  $\rho_{SU(4)}^*$ , and the variation of  $L_S^*$  goes into  $\rho_{SU(2,2)}$ . The precise definition of this infinitesimal variation goes as follows. Take four complex linearly independent vectors  $e_1, e_2, e_3, e_4$  in  $L_A \oplus L_S^*$ , so that  $L_A \oplus L_S^*$  is generated by  $e_1, e_2, e_3, e_4$ . Given  $\Psi_{++}$  of the form (40) we define the variation

$$\delta e_i = (\phi, e_i) \tilde{\eta} - (\eta, e_i) \tilde{\phi} \quad (42)$$

The plane generated by the four vectors  $e_1 + \delta e_1, e_2 + \delta e_2, e_3 + \delta e_3, e_4 + \delta e_4$  is the infinitesimal variation of  $L_A \oplus L_S^*$ .

We have the following picture. The null-surface is a collection of the light rays. Each light ray defines an equator in  $AdS_5$  or equivalently a 2-plane  $L_A \subset \rho_{SU(2,2)}$ , and an equator in  $S^5$  or equivalently a 2-plane  $L_S^* \subset \rho_{SU(4)}^*$ . Turning on  $\theta^I$  corresponds to the deformation of  $L_A \oplus L_S^*$  inside  $\rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$ , so that  $L_A$  is not entirely inside of  $\rho_{SU(2,2)}$  and  $L_S^*$  is not entirely in  $\rho_{SU(4)}^*$ . In other words, while a “purely bosonic” null ray parametrizes a pair of 2-planes

$$(L_A \subset \rho_{SU(2,2)}, L_S^* \subset \rho_{SU(4)}^*)$$

a null ray with  $\theta^I$  parametrizes a 4-plane

$$L_A \oplus L_S^* \subset \rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$$

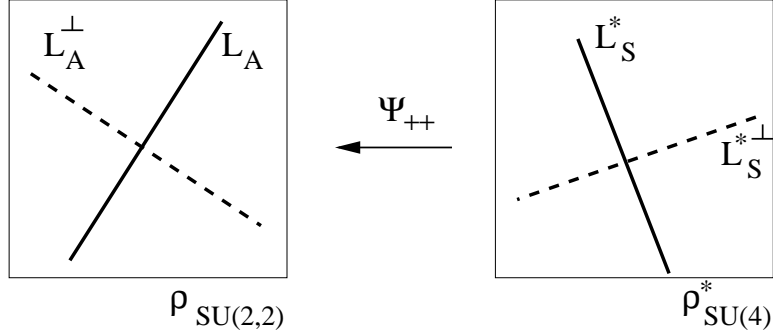


Figure 1: Eq. (40) shows that  $\Psi_{++} \in \rho_{SU(2,2)} \otimes \rho_{SU(4)}$  defines a linear map  $\Psi_{++} : \rho_{SU(4)}^* \rightarrow \rho_{SU(2,2)}$  such that  $L_S^*$  goes into  $L_A^\perp$  and  $L_S^{*\perp}$  goes into  $L_A$ . Therefore  $(\Psi_{++}, \Psi_{++}^*)$  defines a linear automorphism of  $\rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$  and therefore an infinitesimal variation of the plane  $L_A \oplus L_S^* \subset \rho_{SU(2,2)} \oplus \rho_{SU(4)}^*$ .

without the constraints that  $L_A$  belongs to  $\rho_{SU(2,2)}$  or  $L_S^*$  belongs to  $\rho_{SU(4)}^*$ . But we also have to take into account that  $\theta^I$  are odd coordinates. This can be done by declaring  $\rho_{SU(2,2)}$  an “even” space and  $\rho_{SU(4)}^*$  an “odd” space. The total space is now  $\rho_{SU(2,2)} \oplus \Pi \rho_{SU(4)}^*$  where  $\Pi$  means that the vector space is considered odd. And our 4-plane  $L_A \oplus L_S^*$  is actually  $L_A \oplus \Pi L_S^*$ . We are embedding the complex space of dimension  $(2|2)$  in the complex space of dimension  $(4|4)$ :

$$L_A \oplus \Pi L_S^* \subset \rho_{SU(2,2)} \oplus \Pi \rho_{SU(4)}^* \quad (43)$$

This means that turning on the fermionic degrees of freedom replaces the product of two ordinary Grassmanians with the super-Grassmanian:

$$\frac{SU(2,2)}{S(U(2) \times U(2))} \times \frac{SU(4)}{S(U(2) \times U(2))} \rightarrow \frac{U(2,2|4)}{U(2|2) \times U(2|2)} \quad (44)$$

Since  $L_A \oplus \Pi L_S^*$  is a complex space, the super-Grassmanian is a complex supermanifold. Eq. (42) implies that  $\tilde{\eta} \otimes \phi$  and  $\tilde{\phi} \otimes \eta$  are holomorphic coordinates. Therefore the complex structure acts on  $\Psi_{++}$  as follows:

$$I \cdot \Psi_{++} = I \cdot (\tilde{\eta} \otimes \phi + \eta^* \otimes \tilde{\phi}^*) = (i\tilde{\eta} \otimes \phi - i\eta^* \otimes \tilde{\phi}^*) = \widehat{\partial_\tau x_A \Gamma^A} \Psi_{++} \quad (45)$$

Super-Grassmanians are discussed in Chapter 4 of [36].

### 3.3 The Hamiltonian system.

The space of parametrized null surfaces comes with a natural Hamiltonian dynamics. The Hamiltonian as a functional of a contour in the coset space is equal to the action of the particle, which has this contour as a trajectory. The symplectic form is the Kahler form on the coset integrated over the contour. (It is obtained as a limit of the natural symplectic form on the space of the classical string worldsheets.) With the fermionic degrees of freedom turned on, the Hamiltonian is the action of the particle moving on the super-Grassmanian (44). The metric on this space is fixed by the supersymmetry, and the symplectic form follows from the symplectic form of the string worldsheet theory. The action of the string worldsheet theory is:

$$\frac{1}{\epsilon} \int d\tau d\sigma \left\{ (\partial_\tau x)^2 - \epsilon^2 (\partial_\sigma x)^2 + (\overline{\theta^1}, \Gamma^A \widehat{\partial_+ x} D_- \theta^1) + (\overline{\theta^2}, \Gamma^A \widehat{\partial_- x} D_+ \theta^2) \right\} \quad (46)$$

To compute the symplectic form at the leading order in  $\frac{1}{\epsilon}$  we substitute for  $x$  and  $\theta$  a parametrized null-surface. We get for the symplectic form<sup>5</sup>:

$$\omega = \frac{1}{\epsilon} d\phi \wedge dE + \frac{1}{\epsilon} \int d\sigma \left[ (\overline{dY} \wedge dY) + (\overline{dZ} \wedge dZ) + \left( d \left( \overline{\Psi_{++}} [\widehat{Y}, \widehat{Y}] \right) \wedge d\Psi_{++} \right) \right] \quad (47)$$

where  $\phi$  is the relative phase of  $Y$  and  $Z$ , and  $E$  is its conjugate variable (see [12]). This expression is written in the gauge  $(\overline{Y}, \partial_\sigma Y) = (\overline{Z}, \partial_\sigma Z) = \text{const}$  and with the condition (39).

A surprising feature of the Hamiltonian is that it contains a fermionic bilinear with two derivatives. Therefore the evolution equation for the fermion should be of the form

$$\partial_t \Psi_{++} = \partial_\sigma^2 \Psi_{++} \quad (48)$$

In the Metsaev-Tseytlin action the fermion derivatives entered only in the combinations of the form  $\theta d\theta$ . Therefore one would expect equations of motion of the form  $\partial_t \Psi_{++} = \partial_\sigma \Psi_{++}$  rather than (48). But it turns out that after taking the average of the equation of motion over the period the first derivative  $\partial_\sigma \Psi_{++}$  is replaced by the second derivative. Indeed, let us

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<sup>5</sup>The relative sign of  $|dY|^2$  and  $|dZ|^2$  is different from what we had in (1) because in this section we are using the “mostly plus” convention for the metric following [34]. With this convention  $|dY|^2 = -|dY_{-1}|^2 - |dY_0|^2 + \sum_{i=1}^4 |dY_i|^2$  and  $|dZ|^2 = \sum_{I=1}^6 |dZ_I|^2$ .



consider the equation for fermions:

$$\partial_\tau \Psi_{++} - \epsilon \Gamma^A \Gamma^S (\partial_\sigma \Psi_{++})^* = 0 \quad (49)$$

This equations looks suspicious, because the  $\tau$ -derivative of  $\Psi_{++}$  is of the order  $\epsilon$  rather than  $\epsilon^2$ . The slow evolution should be on the time scale  $\Delta\tau \sim \frac{1}{\epsilon^2}$ , not  $\frac{1}{\epsilon}$ . But remember that we have to take the average over the period. If we neglect the terms of the order  $\epsilon^2$  and higher, we will get

$$\Psi(\tau + 2\pi, \sigma) - \Psi(\tau, \sigma) = -2\pi\epsilon \langle \Gamma^A \Gamma^S \rangle (\partial_\sigma \Psi_{++})^* \quad (50)$$

Here  $\langle \Gamma^A \Gamma^S \rangle$  means the average of  $\Gamma^A \Gamma^S$  over the period:

$$\langle \Gamma^A \Gamma^S \rangle = \frac{1}{2} \left( \Gamma^A \Gamma^S + \widehat{\partial_\tau x_A} \widehat{\partial_\tau x_S} \right) \quad (51)$$

But the image of this operator is a kappa-symmetry (37):

$$\left( \Gamma^A \Gamma^S + \widehat{\partial_\tau x_A} \widehat{\partial_\tau x_S} \right) (\partial_\sigma \Psi_{++})^* = (\widehat{\partial_\tau x_A} \Gamma^A + \widehat{\partial_\tau x_S} \Gamma^S) \widehat{\partial_\tau x_A} \Gamma^S (\partial_\sigma \Psi_{++})^*$$

Therefore the slow evolution of  $\Psi_{++}$  in the order  $\epsilon$  is trivial. Let us compute the order  $\epsilon^2$ . To simplify the calculations, we will assume that  $\partial_\sigma x = 0$ . The variation of  $\Psi$  over the period is:

$$\begin{aligned} \Psi(\tau + 2\pi, \sigma) - \Psi(\tau, \sigma) &= \\ &= -\epsilon^2 \int_0^{2\pi} d\tau' \int_0^\tau d\tau'' \Gamma^A(\tau) \Gamma^S(\tau) \Gamma^A(\tau') \Gamma^S(\tau') \partial_\sigma^2 \Psi_{++} \end{aligned}$$

Notice that

$$\begin{aligned} \Gamma^A(\tau) \Gamma^A(\tau') &= -\cos(\tau - \tau') + \sin(\tau - \tau') \widehat{\partial_\tau x_A} \Gamma^A \\ \Gamma^S(\tau) \Gamma^S(\tau') &= \cos(\tau - \tau') + \sin(\tau - \tau') \widehat{\partial_\tau x_S} \Gamma^S \end{aligned}$$

After taking the integrals we get:

$$\begin{aligned} \Psi_{++}(\tau + 2\pi, \sigma) - \Psi_{++}(\tau, \sigma) &= \\ &= \epsilon^2 \left[ \pi^2 (1 - \widehat{\partial_\tau x_A} \Gamma^A \widehat{\partial_\tau x_S} \Gamma^S) - \frac{\pi}{2} (\widehat{\partial_\tau x_A} \Gamma^A - \widehat{\partial_\tau x_S} \Gamma^S) \right] \partial_\sigma^2 \Psi_{++} \end{aligned}$$

The first term in the square brackets is again a kappa-symmetry (37). The second term is an operator constant on the light ray, multiplying the second

derivative of  $\Psi_{++}$ . Now we can introduce the slow time  $t = \epsilon^2 \tau$  and write down the equation for the slow evolution:

$$\partial_t \Psi_{++} = \frac{1}{2} \widehat{\partial_\tau x_A} \Gamma^A \partial_\sigma^2 \Psi_{++} \quad (52)$$

The structure of this equation implies that the Hamiltonian of the slow evolution is of the form

$$H = \int d\sigma \left[ (\partial_\sigma \bar{Y}, \partial_\sigma Y) + (\partial_\sigma \bar{Z}, \partial_\sigma Z) + (\partial_\sigma \bar{\Psi}_{++}, \partial_\sigma \Psi_{++}) \right] \quad (53)$$

We should stress that this expression for the Hamiltonian is valid only in the quadratic order in  $\Psi_{++}$ . The difference in the structure of the fermionic term in (47) and (53) is related to the action of the complex structure on  $\Psi_{++}$ , see Eq. (45).

The terms of the higher order in  $\theta$  should be fixed by the supersymmetry. In fact, the quadratic terms are also fixed by the superconformal symmetry and locality. Still, we think it is a nontrivial fact that the moduli space of null-surfaces in AdS times a sphere has a natural structure of the Hamiltonian system, and that the degrees of freedom are roughly the same as needed to parametrize the single trace operators composed of the large number of elementary fields. And if it is possible to understand the higher loop dynamics along the lines of [8, 9, 10], it is very unlikely that it would be also fixed by the superconformal symmetry.

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